1 Deductive Reasoning and Logical Connectives

As we have seen, proofs play a central role in mathematics and they are based on deductive reasoning. Facts (or statements) can be represented using Boolean variables, i.e., the values of variables can only be *true* or *false* but not both.

Definition 1 *A* **statement** *(or* **proposition***) is a sentence that is true or false* but not both*.*

Example 1 *Examples of deductive reasoning:*

1. *I will have dinner either with Jody or with Ann.*
   Jody is out of town.
   Therefore, I will have dinner with Ann.

2. *If today is Wednesday, then I have to go to class.*
   Today is Wednesday.
   Therefore, I have to go to class.

3. *All classes are held either on Mondays and Wednesdays or on Tuesdays and Thursdays.*
   There is no Discrete Math course on Tuesdays.
   Therefore, Discrete Math course is held on Mondays and Wednesdays.

Definition 2 (premises and conclusion) *A* conclusion *is arrived from assumptions that some statements, called* premises*, are true. An argument being* valid *means that if the premises are all true, then the conclusion is also true.*

Example 2 *An example of an invalid deductive reasoning:*
Either 1+1=3 or 1+1=5.
It is not the case that 1+1=3.
Therefore, 1+1=5.

$p \vee q$	p or q
$\neg p$	not p
$\therefore q$	Therefore, q

If p and q are statement variables, then complicated logical expressions (compound statements) related to p and q can be built from *logical connectives.* First three basic connective symbols are:

example	meaning	
$p \wedge q$	p and q	conjunction
$p \vee q$	p or q	disjunction
$\neg p$	not p	negation

Example 3 *Analyze the logical forms of the following statements:*

1. *Either I go to the store, or we're out of vegitables.*

2. *Steve is happy and John is not happy.*

3. *Either Bill is at work and Jane isn't, or Jane is at work and Bill isn't.*

4. *It is not hot but it is sunny.*

5. *It is neither hot nor sunny.*

6. *Let p, q, and r synbolize "$0 < x$", "$x < 3$", and "$x = 3$", respectively. Write the following inequalities symbolically:*

   (a) $x \leq 3$

   (b) $0 < x < 3$

   (c) $0 \leq x < 3$

Statements such as $p\neg \vee q$, $p \wedge \times q$, and $p\neg q$ all not "grammatical" expressions. We only consider *well-formed formulas* or just *formulas*.

2 Truth Tables

An argument is valid if the premises cannot all be true without the conclusion being true as well. Thus, to understand how words such as *and, or*, and *not* affect the validity of arguments, we must see how they contribute to the truth or falsity of statements containing them.

Definition 3 *If p and q are statement variables, then*

- *$p \wedge q$ is true when, and only when, both p and q are true; If either p or q is false, or if both are false, $p \wedge q$ is false.*
- *$p \vee q$ is true when either p is true, or q is true, or both p and q are true. It is false only when both p and q are false.*
- *$\neg p$ is false if p is true, and it is true if p is false.*

Example 4 *Make Truth tables for the following formulas:*

1. $\neg p \vee q$
2. $(s \vee g) \wedge (\neg s \vee \neg g)$

Example 5 *Determine whether the following arguments are valid.*

1. *Either John isn't stupid and he is lazy, or he's stupid.*
   John is stupid.
   Therefore, John isn't lazy.

2. *The butler and the cook are not both innocent.*
   Either the butler is lying or the cook is innocent.
   Therefore, the butler is either lying or guilty.

Testing an Argument Form for Validity

1. Analyze the logical forms of statements and identify the premises and conclusion.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. If there is a row where all the premises are true and the conclusion is false, then the argument is invalid. Otherwise, it is valid. That is, if in every row where all the premises are true, the conclusion is also true, then the argument is valid.

3 Logical Equivalences

Example 6 *Which of these formulas are equivalent?*

$$\neg(p \wedge q) \qquad \neg p \wedge \neg q \qquad \neg p \vee \neg q$$

In this example, using truth table to determine equivalence, one may discover De Morgan's laws.
Suppose that

- p*: The Yankees won last night.*
- q*: The Red Sox won last night.*
- $\neg(p \wedge q)$*: The Yankees and the Red Sox did not both win last night.*
- $\neg p \vee \neg q$*: Either the Yankees or the Red Sox lost last night.*
- $\neg p \wedge \neg q$*: The Yankees and the Red Sox both lost last night.*

Example 7 *Write negations for each of the following statements:*

- *John is 6 feet tall and he weighs at least 200 pounds.*
- *The bus was late or Tom's watch was slow.*
- $x \not< 2$ *where* x *is a real number.*
- *Jim is tall and Jim is thin.*

Note: For the last example, "Jim is tall and Jim is thin" can be written more compactly as "Jim is tall and thin". The negation of it may be written as "Jim is not tall and thin". This is rather confusing. However, one has to remember that in formal logic the words and *and* or *are allowed only between complete statements, not between sentence fragments. One lesson to be learned from this example is that when you apply De Morgan's laws, you must have complete statements on either side of each* and *and on either side of each* or.

Example 8 *Use De Morgan's laws to write the negation of* $1 < x \leq 4$.

Definition 4 (tautology and contradiction) *Statements that are always true, such as* $p \vee \neg p$*, are called* tautologies. *Statements that are always false, such as* $p \wedge \neg p$*, are called* contradictions.

How to check whether a statement is a tautology or contradiction?

Example 9 *Are these statements tautologies, contradictions, or neither?*

$$p \vee (q \vee \neg p) \qquad p \wedge \neg(q \vee \neg q) \qquad p \vee \neg(q \vee \neg q)$$

Theorem 1 *Given any statement variables* $p, q,$ *and* r*, a tautology* $\mathbf{t}$ *and a contradiction* $\mathbf{c}$*, the following logical equivalences hold.*

Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws:	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
Negation laws:	$p \vee \neg p \equiv \mathbf{t}$	$p \wedge \neg p \equiv \mathbf{c}$
Double negative law:	$\neg(\neg p) \equiv p$	
Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal bound laws:	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
De Morgan's laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of $\mathbf{t}$ *and* $\mathbf{c}$*:*	$\neg \mathbf{t} \equiv \mathbf{c}$	$\neg \mathbf{t} \equiv \mathbf{c}$

Example 10 *Find simpler formulas equivalent to these formulas:*

1. $\neg(q \vee \neg p) \vee p$
2. $\neg(\neg p \wedge q) \wedge (p \vee q)$

4 Variables and Sets

In mathematical reasoning, we often use letters, called *variables*, as a placeholder to represent objects, and make statements about these objects. For example, consider the statement:

No matter what number might be chosen, if it is greater than 3, then its square is greater 9.

We introduce a variable to replace the words referring to the number:

No matter what number n might be chosen, if n is greater than 3, then n^2 is greater than 9.

If a variable is used to stand for an object, we may be interested in talking about the properties of the object. For example, to express that a number is prime, we can use x to represent the number. To express the statement "x is prime", so far we have been using some Boolean variable such as p. However, such a notation does not stress that p is a statement *about* x. Therefore, we sometimes use $p(x)$ to represent the statement, implying that the statement is about x. Such a notation makes it easy to talk about substituting some number for x in the statement. For instance, $p(5)$ represents the statement "5 is prime" and $p(a+b)$ means "$a+b$ is prime".

Note that a variable (placeholder for objects) here is different from Boolean variables (statement variables) which are meant to be statements. We usually use x, y, z for placeholders for objects and p, q, r, s, t for Boolean variables.

Example 11 *Analyze the logical forms of the following statements:*

1. *x is a prime number, and either y or z is divisible by x.*
2. *x and y are people and x likes y, but y doesn't like x.*

Note that in the previous examples, one can not describe the statement as being true of false without evaluating the value of variables! For example, if $p(x)$ means "x is prime", then $p(x)$ would be true if x is 3 but false if x is 4. Before we get into the truth sets for statements involving variables, it will be helpful to review some basic definitions from set theory.

Definition 5 (set) *A* set *is a collection of objects. An object in the collection is called an* element *of the set. We use the symbol* $\in$ *to mean* is an element of*.*

For example if A stands for the set $\{1, 2, 3\}$, then $2 \in A$ and $5 \notin A$. To define a set, we have the following notations:

- Set-Roster notation: $\{1, 2, 3\}$, $A = \{2, 4, 6, ...\}$
- Set-Builder notation: $\{x \in \mathbb{Z} \,|\, -3 < x < 8\}$, $B = \{x \,|\, x \text{ is a prime number}\}$

Rewrite set A above using the Set-Builder notation, we have $A = \{x \,|\, x \text{ is a positive even number}\}$. The definition of above set B is read "B is the set of all x such that x is a prime number" where the statement "x is a prime number" can be though of as an *elementhood test*.

If a set is defined using Set-Builder notation, we use elementhood test to check whether something is an element of the set of not. For example, $9 \notin \{x \in \mathbb{Z} \,|\, -3 < x < 8\}$. Note that for any integer y, $y \in \{x \in \mathbb{Z} \,|\, -3 < x < 8\}$ talks about y, but not x! To determine whether $y \in \{x \in \mathbb{Z} \,|\, -3 < x < 8\}$, one needs to know what y's value is, but not what x is. Therefore, in the statement $y \in \{x \in \mathbb{Z} \,|\, -3 < x < 8\}$, we say that y is a *free variable* and x is a *bound variable*. The free variables in a statement stand for objects that the statement is about. Different values of a free variable may change the truth value of a statement. Note that the meaning of a statement will not be changed if a bound variable is replaced with another variable. For example, $y \in \{x \in \mathbb{Z} \,|\, -3 < x < 8\}$ and $y \in \{z \in \mathbb{Z} \,|\, -3 < z < 8\}$ mean the same thing.

In statement $-3 < x < 8$, variable x is a free variable . Only when $-3 < x < 8$ is used inside the elementhood test notation will x become a bound variable. The notation $\{x \,|\, ...\}$ *binds* the variable x. In general, $y \in \{x \,|\, p(x)\}$ means the same thing as $p(y)$, which is a statement about y (but not x).

Also note that the expression $\{x \,|\, p(x)\}$ is not a statement, but an expression describing the name of a set.

Example 12 *What do these statements mean? What are the free variables in each statement?*

1. $a - b \notin \{x \,|\, x \textit{ is an even number}\}$.
2. $y \in \{x \,|\, x \textit{ is divisible by } w\}$.
3. $3 \in \{w \,|\, 12 \notin \{x \,|\, x \textit{ is divisible by } w\}\}$.

Definition 6 (Truth Set) *The set of all values of x that make the statement $p(x)$ true is called the* truth set *of $p(x)$. That is, the truth set of $p(x)$ is the set $\{x \,|\, p(x)\}$.*

Example 13 *What is the truth set of the statement "n is an even prime number"?*

In general, if A is the truth set of $p(x)$, then to say that $y \in A$ means the same thing as saying $p(y)$.

Definition 7 (Universe of Discourse) *The set of all possible values of a free variable in a statement is called the* universe of discourse *for the statement, and we say that the variables* ranges over *this universe.*

Here are some examples of sets with fixed names:

- $\mathbb{R} = \{x \,|\, x \text{ is a real number}\}$. A *real* number is any number on the number line.

- $\mathbb{Q} = \{x \,|\, x \text{ is a rational number}\}$. A *rational* number is a number than can be written as a fraction a/b where a and b are integers.
- $\mathbb{Z} = \{x \,|\, x \text{ is an integer}\} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$.
- $\mathbb{N} = \{x \,|\, x \text{ is a natural number}\} = \{0, 1, 2, 3, ...\}$.

Different universes may cause the truth set to be different for the same statement. For example, consider the statement $x^2 = 4$. Its truth set over integers is $\{x \in \mathbb{Z} \,|\, x^2 = 4\} = \{-2, 2\}$ and its truth set over natual numbers is $\{x \in \mathbb{N} \,|\, x^2 = 4\} = \{2\}$. Therefore, when one considers the universe of discourse, checking whether a value is an element of the set, besides the elementhood test, one also needs to check whether the value is an element of the universe.

In general, $y \in \{x \in A \,|\, p(x)\}$ means the same thing as $y \in A \wedge p(y)$.

Example 14 *Simplify the statement* $4 \in \{x \in \mathbb{R}^- \,|\, 13 - 2x > 1\}$ *where* $\mathbb{R}^-$ *is the set of all negative real numbers.*

Given a universe U, the statement $p(x) \vee \neg p(x)$ is a tautology, and hence true for every $x \in U$. Therefore, the truth set of the statement $p(x) \vee \neg p(x)$ is U. Similarly, the truth set for statements that are contradictions has no elements. Such truth set is called the *empty set*, or the *null set.* and is often denoted $\varnothing$. Note that $\{\varnothing\}$ is different from $\{\} = \varnothing$.

Exercise 1 *Analyze the logical forms of the following statements:*

1. *3 is a common divisor of 6, 9, and 15.*
2. *x and y are natural numbers, and exactly one of them is prime.*

Exercise 2 *Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.*

1. $3 \in \{x \in \mathbb{R} \,|\, 13 - 2x > 1\}$.
2. $5 \notin \{x \in \mathbb{R} \,|\, 13 - 2x > c\}$.

Example 15 *Is* 0.12121212... *a rational number (where the digits 12 are assumed to be repeat forever)?*

Example 16 *Prove that the sum of any two rational numbers is rational.*

5 Operations on Sets

Definition 8 (union, intersection, difference, and complement) *Let A and B be subsets of a universal set U.*

- *The* union *of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A or B.*
- *The* intersection *of A and B, denoted $A \cap B$, is the set of all elements that are common to both A and B.*

- *The* difference *of B minus A (or* relative complement *of A in B), denoted $B \setminus A$, is the set of all elements that are in B and not A.*

- *The* complement *of A, denoted A^c, is the set of all elements in U that are not in A.*

$$\begin{aligned} A \cup B &= \{x \in U \mid x \in A \textit{ or } x \in B\}, \\ A \cap B &= \{x \in U \mid x \in A \textit{ and } x \in B\}, \\ B \setminus A &= \{x \in U \mid x \in B \textit{ and } x \notin A\}, \\ A^c &= \{x \in U \mid x \notin A\}. \end{aligned}$$

Venn diagram representation...

Example 17 *Let the universe $U = \{a, b, c, d, e, f, g\}$, $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find $A \cup B$, $A \cap B$, $B \setminus A$, and A^c.*

Notation 1 *Given real numbers a and b with $a \leq b$:*

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\} \qquad [a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

$$(a, b] = \{x \in R | a < x \leq b\} \qquad [a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}.$$

The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:

$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\} \qquad (-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$

$$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\} \qquad [-\infty, b) = \{x \in \mathbb{R} \mid x \leq b\}.$$

Example 18 *Let the universe be the set $\mathbb{R}$ of all real numbers, $A = (-1, 0] = \{x \in \mathbb{R} \mid -1 < x \leq 0\}$, and $B = [0, 1) = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$. Find $A \cup B$, $A \cap B$, $B \setminus A$, and A^c.*

Recall that if A is the truth set of a statement $p(x)$ and B is the truth set of $q(x)$, then $x \in A$ means the same thing as $p(x)$ and $x \in B$ means the same thing as $q(x)$. Thus, the truth set of $p(x) \wedge q(x)$ is $\{x \mid p(x) \wedge q(x)\} = \{x \mid x \in A \wedge x \in B\} = A \cap B$.

These observations about truth sets illustrate the fact that the set theory operations $\cap, \cup$, and $\setminus$ are related to the logical connectives $\wedge, \vee$, and $\neg$. However, it is important to remember that although the set theory operations and logical connectives are related, they are not interchangeable. The logical connectives can only be used to combine *statements*, whereas the set theory operations must be used to combine *sets*. Therefore, expressions such as $A \wedge B$ or $p(x) \cap q(x)$ are completely meaningless and should never be used.

Example 19 *Analyze the logical forms of the following statements:*

1. $x \in A \cap (B \cup C)$
2. $x \in A \setminus (B \cap C)$
3. $x \in (A \cap B) \cup (A \cap C)$

Exercise 3 *Analyze the logical forms of statements $x \in (A \cup B) \setminus (A \cap B)$ and $x \in (A \setminus B) \cup (B \setminus A)$. Are these two statements equivalent? If they are, show why they are equivalent.*

Definition 9 (subset and disjoint set) *Suppose A and B are sets. Then A is a* subset *of B, denoted by $A \subseteq B$, if every element of A is also an element of B. Sets A and B are said to be* disjoint *if they have no elements in common. In other words $A \cap B = \varnothing$, i.e., the set of elements A and B have in common is the empty set.*

Example 20 *Consider sets $A = \{a, c\}, B = \{a, b, c, d, e\}$, and $C = \{e, f\}$. Then since the two elements of A, a and c, are both in B, we have $A \subseteq B$. Also, $A \cap C = \varnothing$, so A and C are disjoint.*

Example 21 *Prove that for any sets A and B, $A \cap B$ and $A \setminus B$ are disjoint. For any sets A and B, $A \cap B$ and $A \setminus B$ are disjoint.*

Example 22 *Prove that for any sets A and B, $(A \cup B) \setminus B \subseteq A$.*

Exercise 4 *For each of the following sets, write out (using logical symbols) what it means for an object x to be an element of the set. Then determine which of these sets must be equal to each other by determining which statements are equivalent.*

1. $(A \setminus B) \setminus C$.
2. $A \setminus (B \setminus C)$.
3. $(A \setminus B) \cup (A \cap C)$.
4. $(A \setminus B) \cap (A \setminus C)$.
5. $A \setminus (B \cup C)$.

Definition 10 (Unions and Intersections of an Indexed Collection of Sets) *Given sets A_0, A_1, A_2, . . . that are subsets of a universal set U and given a nonnegative integer n,*

$$\begin{aligned}
\bigcup_{i=0}^{n} A_i &= \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, ..., n\} = A_0 \cup A_1 \cup ... \cup A_n \\
\bigcup_{i=0}^{\infty} A_i &= \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\} \\
\bigcap_{i=0}^{n} A_i &= \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, ..., n\} = A_0 \cap A_1 \cap ... \cap A_n \\
\bigcap_{i=0}^{\infty} A_i &= \{x \in U \mid x \in A_i \text{ for all nonnegative integer } i\}
\end{aligned}$$

Example 23 *For each positive integer i, let $A_i = \{x \in \mathbb{R} \mid -\frac{1}{i} < x < \frac{1}{i}\} = (-\frac{1}{i}, \frac{1}{i})$.*

1. *Find $A_1 \cup A_2 \cup A_3$ and $A_1 \cap A_2 \cap A_3$.*
2. *Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.*

6 The Conditional and Biconditional Statements

6.1 The Conditional Connective $\rightarrow$

We have seen the following reasoning form in Example 1 part 1 and 3 from Section 1:

$p \vee q$	p or q
$\neg p$	not p
$\therefore q$	Therefore, q

What about the reasoning in the second argument? Recall that the argument went like this:

If today is Tuesday, then I have to go to class.
Today is Tuesday.
Therefore, I have to go to class.

What makes this argument valid?

For statements of the form "If p then q", we represent it by $p \rightarrow q$, which is called a *conditional statement*, where p is the *antecedent/hypothesis* and q is the *consequent/conclusion*. Let p stand for the statement "today is Tuesday" and q for "I have to go to class", then the logical form of the argument is as follows:

$p \rightarrow q$	if p then q
p	p
$\therefore q$	Therefore, q

Example 24 *Analyze the logical forms of the following statements:*

1. *If it's raining and I don't have my umbrella, then I'll get wet.*
2. *If Mary did her homework, then the teacher won't collect it, and if she didn't, then the teacher will ask her to do it on the board.*

6.2 Truth Table for $\rightarrow$

How to build the truth table for the connective $\rightarrow$? Since $p \rightarrow q$ is supposed to mean that if p is true then q is also true, it is clear that that if p is true and q is false then $p \rightarrow q$ is false. If p is true and q is also true, then it seems reasonable to say that $p \rightarrow q$ is true. What about two other cases?

Consider the statement "If $x > 2$ then $x^2 > 4$".

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

p	q	$\neg p \vee q$
F	F	T
F	T	T
T	F	F
T	T	T

6.3 $p \rightarrow q$ and $\neg p \vee q$

In Example 4, we've seen the truth table for $\neg p \vee q$. It appears that the truth table for $\neg p \vee p$ and $p \rightarrow q$ are the same. Is it consistent with the way words "if... then..." are used? It turned out that it is. Example...

6.4 $p \to q$ and $\neg(p \wedge \neg q)$

$$\begin{array}{lll} & p \to q & \\ \text{is equivalent to} & \neg p \vee q & \\ \text{is equivalent to} & \neg p \vee \neg\neg q & \text{Double negation law} \\ \text{is equivalent to} & \neg(p \wedge \neg q) & \text{De Morgan's law} \end{array}$$

Thus, $p \to q$ is also equivalent to $\neg(p \wedge \neg q)$.

Consider an example "If it's going to rain, then I'll take my umbrella."

Conditional laws		
$p \to q$	is equivalent to	$\neg p \vee q$
$p \to q$	is equivalent to	$\neg(p \wedge \neg q)$

Example 25 *Write negations for each of the following statements:*

1. *If my car is in the repair shop, then I cannot get to class.*
2. *If Sara lives in Athens, then she lives in Greece.*

6.5 Another Argument for $\to$

Consider again the truth table for $\to$

#	p	q	$p \to q$	p	q
1	F	F	T	F	F
2	F	T	T	F	T
3	T	F	F	T	F
4	T	T	T	T	T

and the following two cases:

- Make $p \to q$ false in line #1 of the table. Take $p \to q$ and p as premises and q as conclusion for the reasoning form. Then the value of conclusion q does not depend on the premise p. This means that from the single premise $p \to q$, we could infer that q must be true.
- Make $p \to q$ false in line #2 of the table. Take $p \to q$ and q as premises. We would obtain the following reasoning form that is valid!

  $$\begin{array}{c} p \to q \\ q \\ \therefore p \end{array}$$

  However, this should **not** be considered a valid form of reasoning. Example...

6.6 Converse, Inverse, and Contrapositive

- The *converse* of $p \to q$ is $q \to p$.
- The *inverse* of $p \to q$ is $\neg p \to \neg q$.
- The *contrapositive* of $p \to q$ is $\neg q \to \neg p$.

Example 26 *Write each of the following statements in its converse, inverse and contrapositive forms:*

1. *If Howard can swim across the lake, then Howard can swim to the island.*
2. *If today is Easter, then tomorrow is Monday.*

Facts
A conditional statement and its converse are *not* logically equivalent.
A conditional statement and its inverse are *not* logically equivalent.
The converse and the inverse of a conditional statement are logically equivalent to each other.
A conditional statement is logically equivalent to its contrapositive.

Example 27 *Analyze the logical forms of the following statements:*

1. *If at least ten people are there, then the lecture will be given.*
2. *The lecture will be given only if at least ten people are there.*
3. *The lecture will be given if at least ten people are there.*
4. *Having at least ten people there is a sufficient condition for the lecture being given.*
5. *Having at least ten people there is a necessary condition for the lecture being given.*

If r and s are statements:		
r is a **sufficient condition** for s	means	“if r then s”
r is a **necessary condition** for s	means	“if not r then not s”, equivalent to “if s then r”

Contrapositive law		
$p \to q$	is equivalent to	$\neg q \to \neg p$

Exercise 5 *Analyze the logical forms of the following statements and find out which of the following statements are equivalent.*

1. *If it’s either raining or snowing, then the game has been canceled.*
2. *If the game hasn’t been canceled, then it’s not raining and it’s not snowing.*
3. *If the game has been canceled, then it’s either raining or snowing.*
4. *If it’s raining then the game has been canceled, and if it’s snowing then the game has been canceled.*
5. *If it’s neither raining nor snowing, then the game hasn’t been canceled.*

6.7 Biconditional Statement

Definition 11 *Given statement variables p and q, the* biconditional *of p and q is "p if q and p only if q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words if and only if are sometimes abbreviated* iff.

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Example 28 *Analyze the logical forms of the following statements:*

1. *The game will be canceled iff it's either raining or snowing.*
2. *Having at least ten people there is a necessary and sufficient condition for the lecture being given.*
3. *If I go to the store then we have some eggs, and if I don't then we don't.*

6.8 Order of Operation for Logical Operators

		Order of Operation for Logical Operators
1.	$\neg$	Evaluate negations first.
2.	$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3.	$\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

7 Proofs and Proof Strategies

In mathematics, a *proof* is a deductive argument to show the correctness of some mathematical statement. Such a mathematical statement is usually in the form of a *theorem* that says that if some assumptions, called the *hypotheses*, of the theorem are true, then some conclusion must also be true. Before the correctness of a theorem is proved, the theorem is often called a *conjecture*. A proof must demonstrate that a statement is *always* true (rather than showing the statement is *sometimes* true). When the hypotheses and conclusion contain free variables, it is understood that these variables can stand for any elements of the universe of discourse. A theorem is correct only when for every element in the universe assigned to the free variable, if the hypotheses are true, then the conclusion is also true. If there is even one value of a free variable in which the hypotheses are true but the conclusion is false, then the theorem is incorrect. Such an instance is called a *counterexample* to the conjecture. In this section, we discuss some proof strategies.

7.1 Direct Proof

Strategy 1: If one of the given assumptions has the form $\neg p$, reexpress this assumption in some other form when possible. Usually it's easier to prove a positive than a negative statement.

Example 29 *Suppose that $A \subseteq B$, $a \in A$, and $a \notin B \setminus C$. Prove that $a \in C$.*
Proof sketch.
Given: $A \subseteq B$, $a \in A$, $a \notin B \setminus C$
Goal: $a \in C$

Strategy 2: If one of the goal has the form $\neg p$, reexpress this assumption in some other form when possible.

Example 30 *Suppose $A \cap C \subseteq B$ and $a \in C$. Prove that $a \notin A \setminus B$.*
Proof sketch.
Given: $A \cap C \subseteq B$ and $a \in C$.
Goal: $a \notin A \setminus B$.
Reexpress the goal:

Strategy 3 (Modus Ponens): If one of the given assumptions has the form $p \to q$, and p is either given or proven to be true, then you can conclude that q is true.

Modus Ponens
$p \to q$
p
$\therefore q$

Example 31 *Suppose a and b are real numbers. Prove that if $0 < a < b$ then $a^2 < b^2$.*

Strategy 4 (Generalization): If the goal is to prove $p \vee q$, it is sufficient to prove only one of p and q.
Strategy 5 (Specialization): (a) If the goal is to prove $p \wedge q$, prove p and q separately. (b) If one given assumption is $p \wedge q$, treat the assumption as two separate assumptions: p and q.
Strategy 6 (Elimination): If a given assumption is of the form $p \vee q$, and $\neg q$ is either given or proven to be true, then you can conclude p.

Elimination (case 1)
$p \vee q$
$\neg q$
$\therefore p$

Elimination (case 2)
$p \vee q$
$\neg p$
$\therefore q$

Strategy 7 (Transitivity): If the goal is to prove $p \to r$, and from the assumptions you can prove $p \to q$ and $q \to r$, then you can conclude $p \to r$.

Transitivity
$p \to q$
$q \to r$
$\therefore p \to r$

Strategy 8 (Proof by Division into Cases): If you are given or already have proven statements of the forms $p \vee q$, $p \to r$, and $q \to r$, then you can conclude r.

Proof by Division into Cases
$p \vee q$
$p \to r$
$q \to r$
$\therefore r$

Example 32 *You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:*

1. *If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.*

2. *If my glasses are on the kitchen table, then I saw them at breakfast.*

3. *I did not see my glasses at breakfast.*

4. *I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.*

5. *If I was reading the newspaper in the living room then my glasses are on the coffee table.*

Where are the glasses?

7.2 Indirect Proof: by Contradiction and Contraposition

Strategy 1 Proof by Contradiction: If one of the given assumptions has the form $\neg p$, then you can produce a contradiction by proving p. Or assume the statement to be proved is false, show that the supposition leads logically to a contradiction, and conclude that the statement to be proved is true.

Example 33 *Prove that if* $x^2 + y = 13$ *and* $y \neq 4$ *then* $x \neq 3$.
Proof sketch.

Suppose that $x^2 + y = 13$ *and* $y \neq 4$.
  Suppose that $x = 3$.
    [Proof of contradiction goes here]
  Therefore, $x \neq 3$.
Hence, if $x^2 + y = 13$ *and* $y \neq 4$ *then* $x \neq 3$.

Strategy 2 Proof by Contraposition (Modus Tollens): If one of the given assumptions has the form $p \to q$, since it is equivalent to $\neg q \to \neg p$, then you can produce a contradiction by proving p.

Modus Tollens
$p \to q$
$\neg q$
$\therefore \neg p$

Example 34 *Suppose* $a, b,$ *and* c *are real numbers and* $a > b$. *Prove that if* $ac \leq bc$ *then* $c \leq 0$.

Example 35 *Suppose that* $A \setminus B \subseteq C \cap D$ *and* $x \in A$. *Prove that if* $x \notin D$ *then* $x \in B$.

Example 36 *Suppose* x *is a real number and* $x \neq 0$. *Prove that if* $\dfrac{\sqrt[3]{x}+5}{x^2+6} = \dfrac{1}{x}$, *then* $x \neq 8$.

Example 37 *Suppose that* $A, B,$ *and* C *are sets, and* $A \setminus B \subseteq C$. *Prove that if* $x \in A \setminus C$ *then* $x \in B$.
Proof sketch.
Given: $A \setminus B \subseteq C$ *and* $x \in A \setminus C$
Goal: $x \in B$
The form of proof is as follows:

Suppose that $x \in A \setminus C$.
  [Proof of $x \in B$ *goes here].*
Therefore, if $x \in A \setminus C$ *then* $x \in B$.

Now our goal has no logical connectives and it is not obvious why the goal follows from the assumptions. Then we should try to prove by contradiction:

Suppose that $x \in A \setminus C$*.*
    Suppose that $x \notin B$*.*
        [Proof of contradiction goes here]
    Therefore, $x \in B$*.*
Therefore, if $x \in A \setminus C$ *then* $x \in B$*.*

Acknowledgement
Much of the content of the notes is based on the following books:

- *Discrete Mathematics with Applications, 4th Edition* by Susanna Epp, Cengage Learning 2010.
- *How to Prove It: A Structured Approach, 2nd Edition* by Daniel J. Velleman, Cambridge University Press 2006